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Optimal linear transmission by loss-insensitive packet encoding ☆

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Abstract

The objective of this paper is to characterize the optimal use of redundancy in transmitting a signal that is encoded in terms of packets of linear coefficients. The signals considered here are vectors in a finite-dimensional real or complex Hilbert space. For the purpose of transmission, these vectors are encoded in a set of linear coefficients that is partitioned in packets of equal size. We investigate how the encoding performance depends on the degree of redundancy it incorporates and on the amount of data-loss when packets are either transmitted perfectly or lost in their entirety. The encoding performance is evaluated in terms of the maximal Euclidean norm of the reconstruction error occurring for the transmission of unit vectors. Our main result is the derivation of error bounds as well as the characterization of optimal encoding when up to three packets are lost.

Keywords: Frames; Codes; Erasures; Error bounds

1. Introduction

The transmission of digital media typically follows a protocol that splits data into a number of packets having a fixed size. When such packets are sent over a network such as the Internet, there is in principle no guarantee of reliability, that is, the contents of each packet may become corrupted in the course of transmission or entire packets may be lost due to buffer overflows. The integrity of the data in each packet is typically protected by some error correction scheme, so for practical purposes one may assume that packets arrive either intact or not at all. The objective of this paper is to find an optimal way of incorporating redundancy in a signal that is transmitted in the form of packets of linear coefficients.

The signal under consideration is a vector in a k-dimensional (real or complex) Hilbert space, which is transmitted in the form of m packets of l linear coefficients. We assume $k < ml$ to allow for the possibility that a significant part of the signal may be recovered without the need for resending when a few packets are lost (or impractically delayed) in the transmission. To quantify performance, we maximize the Euclidean reconstruction error due to a lossy transmission over the set of all unit vectors.

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In practice, it is commonly assumed that losing one packet in the transmission process is rare, and that the occurrence of two lost packets is much less likely. A similar hierarchy of probabilities usually holds for a higher number of lost packets. This motivates the design of transmission protocols following an inductive scheme: We require perfect reconstruction when no data is lost. Among the protocols giving perfect reconstruction, we want to minimize the maximal error in the case of one lost packet. Generally, we continue by choosing among the protocols which are optimal for q losses those performing best for $q + 1$ losses. For an alternative approach, which does not assume a hierarchy of errors, see the so-called maximally robust encoding [16].

The characterization of optimality derived here is a generalization of results by Casazza and Kovačević [3] as well as Holmes and Paulsen [10] on the design of frames for linear encoding. The central assumption of the network models in these previous results was that a *sequence* of vectors is transmitted in the form of their frame coefficients. These coefficients are sent in parallel streams to the receiver, see [8, Example 1.1] and [12]. The design of frames for this task has proved a fascinating subject, in particular in the presence of additional sources of noise, such as additive noise and quantization errors [8,9,12].

When the only possible error is the loss, also referred to as erasure, of one frame coefficient, uniform frames were shown to be optimal [3]. In a following paper [10], a family of so-called *two-uniform* frames was introduced. When they exist, two-uniform frames were demonstrated to be optimal for one and two erasures. Moreover, it was proved that a frame is two-uniform if and only if it is equiangular, which constitutes a family of frames that has been studied independently by Strohmer and Heath [17].

The existence of such frames, over the reals, depends on the existence of a matrix of ± 1's which satisfies certain algebraic equations. Bodmann and Paulsen [2] related the performance of these frames in the presence of higher numbers of erasures to graph theoretic quantities. The construction of two-uniform frames in the complex case was investigated with number-theoretic tools, see [7,13,21], as well as with a numerical scheme [19].

Here, we generalize results on the suppression of errors due to lost *coefficients* in frame-based encoding to the situation when the frame coefficients are partitioned into subsets of equal size and if erasures occur, then the contents of entire *packets* (subsets of coefficients) are set to zero. For related problems, see the work by Oswald on stable space splittings in Hilbert spaces [15] which are equivalent to Sun's notion of g-frames [18], and the concept of frames for subspaces introduced by Casazza and Kutyniok [4]. This concept was applied under the name of fusion frames to distributed processing [5].

The first main result of this paper is the characterization of optimal encoding when at most $q = 1$ packet is lost. We define so-called uniform (m, l, k)-protocols in terms of m uniformly weighted rank-l projections resolving the identity on the k-dimensional Hilbert space containing the vectors to be transmitted. If such uniform protocols exist, then they minimize the worst-case Euclidean reconstruction error for the transmission of unit vectors.

The second main result in this paper specifies the best encoding among uniform protocols when up to $q = 2$ packets are lost. To this end, we define two-uniform (m, l, k)-protocols as those given by a uniformly weighted family of projections resolving the identity which satisfy that the ranges of the projections form so-called equi-isoclinic subspaces. If two-uniform protocols exist, they are optimal for up to two lost packets.

Finally, we derive an optimality condition for two-uniform protocols when up to $q = 3$ packets are lost and compute an error bound for two-uniform protocols that is valid for all $q \in \mathbb{N}$.

This paper is organized as follows. After fixing the notation in Section 2, we discuss the idea of using certain linear maps as linear transmission protocols in Section 3 and introduce a numerical measure for the error when the coded information is partially lost. We derive error bounds and investigate which protocols perform best for up to one, two and three lost packets. Section 4 derives an error bound for two-uniform protocols and any number of lost packets.

Much of this paper was influenced by the very transparent insights Vern Paulsen has shared with me over the last few years. Thanks go to Chris Godsil for pointing out results on equi-isoclinic subspaces and their construction, and to the anonymous referees for the attentive review and detailed corrections.

2. Preliminaries and notation

We begin by introducing the basic definitions and concepts.

Definition 1. Let $\mathcal{H}$ be a real or complex Hilbert space, and let $\{V_j\}_{j\in\mathbb{J}}$ be an at most countable family of linear maps $V_j:\mathcal{H}\to\mathcal{K}$ into a Hilbert space $\mathcal{K}$. We say $\{V_j\}_{j\in\mathbb{J}}$ is a set of *coordinate operators* on $\mathcal{H}$ if they form a *resolution of the identity*

$$\sum_{j\in\mathbb{J}} V_j^* V_j = I.$$

Remark 2. If the dimension of $\mathcal{K}$ is one, then having a resolution of the identity implies that the column vectors $f_j = V_j^*$ form a *Parseval frame*, that is for all $x\in\mathcal{H}$,

$$\sum_{j\in\mathbb{J}} \bigl|\langle x, f_j\rangle\bigr|^2 = \sum_{j\in\mathbb{J}} |V_j x|^2 = \|x\|^2.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the inner product of two vectors, by convention conjugate linear in the second entry if $\mathcal{H}$ is a complex Hilbert space.

We observe that the so-called *analysis operator* V formed by combining the blocks $\{V_j\}_{j\in\mathbb{J}}$ as rows in an isometry

$$V:\mathcal{H}\to\bigoplus_{j\in\mathbb{J}}\mathcal{K},\quad (Vx)_j = V_j x$$

has its adjoint V^* as a left inverse.

A resolution of the identity has also been called a tight g-frame [18]. The notion of g-frames is equivalent to that of stable space splittings of Hilbert spaces [15]. We focus on a special case.

Definition 3. We call a family $\{V_j\}_{j\in\mathbb{J}}$ the coordinate operators of a *weighted projective resolution of the identity* if $\sum_{j\in\mathbb{J}} V_j^*V_j = I$ and for each $j\in\mathbb{J}$, $V_j^*V_j = k_jP_j$ with a nonnegative coefficient k_j and a self-adjoint projection $P_j = P_j^*P_j$.

We call a family of coordinate operators $\{V_j\}$ *uniform* or *equal-norm* provided there is a constant $c>0$ so that the operator norm $\|V_j\| = c$ for all $j\in\mathbb{J}$.

Finally, we say the family $\{V_j\}$ belongs to a *uniformly weighted projective resolution of the identity* if it combines these two properties.

Since each projection can be characterized by its range, one can also think of a weighted projective resolution of the identity as a collection of subspaces with associated weights. The above special case of a resolution of the identity was introduced as Parseval frames for subspaces by Casazza and Kutyniok [4], and later applied under the name of Parseval fusion frames to analyze problems in distributed processing [5].

For the purposes of this paper we will only be concerned with finite dimensional Hilbert spaces and resolutions of the identity for these spaces that involve a sum of finitely many operators indexed by $\mathbb{J}=\{1,2,\ldots,m\}$. When the dimension of $\mathcal{H}$ is k, we identify $\mathcal{H}$ with $\mathbb{R}^k$ or $\mathbb{C}^k$ depending on whether we are dealing with the real or complex case. When we wish to refer to either case, then we will denote the underlying field by $\mathbb{F}$. We will often choose an orthonormal basis and regard vectors as columns and operators as matrices. Whenever we speak of unitary matrices, we mean orthogonal ones in the real case.

We sketch a construction of certain families of uniformly weighted projections resolving the identity. Factoring each projection into coordinate operators provides a corresponding transmission protocol.

Example 4. We first construct examples of weighted projections resolving the identity for which all $\{k_jP_j\}_{j=1}^m$ are rank-one and $m>k$. To this end, we take a self-adjoint rank-k projection $G=G^*G$ on $\mathbb{F}^m$ and factor it as $G=VV^*$, where V is an isometry $V:\mathbb{F}^k\to\mathbb{F}^m$. Now denote the m column vectors of the matrix representation for V^* in the standard basis as $\{f_j\}_{j=1}^m$, choose each P_j to be the projection on the one-dimensional subspace of $\mathbb{F}^k$ spanned by f_j, and let the set of weights be $\{k_j=\|f_j\|^2\}$. It is then straightforward to verify that $\sum_{j=1}^m k_jP_j = I$.

Holmes and Paulsen [10, Remark 1.1] show how rotating amongst the frame vectors $\{f_j\}$ according to [11] makes the weights uniform.

The next example describes how rank-one projections may be used to construct higher-rank weighted projections which resolve the identity.

Example 5. If the Hilbert space is a tensor product $\mathbb{F}^k = \mathbb{F}^s \otimes \mathbb{F}^l$, so $k = sl$, and there is a set of weighted rank-one projections $\{k_j \Pi_j\}_{j=1}^m$ resolving the identity on the first component $\mathbb{F}^s$, then $\{k_j P_j = k_j \Pi_j \otimes I\}$ is a set of weighted rank-l projections resolving the identity on $\mathbb{F}^k$.

If all weights are uniform in the resolution of the identity for the first component, the same holds for the tensor product.

Definition 6. We shall let $\mathcal{V}(m,l,k)$ denote the collection of all families $\{V_j\}_{j=1}^m$ consisting of $m \in \mathbb{N}$ coordinate operators $V_j : \mathcal{H} \to \mathcal{K}$ of maximal rank $l \in \mathbb{N}$ that provide a resolution of the identity for the Hilbert space $\mathcal{H} = \mathbb{F}^k$, $k \in \mathbb{N}$. We call the analysis operator V of such a family $\{V_j\} \in \mathcal{V}(m,l,k)$ an *(m,l,k)-protocol*.

If the family of such coordinate operators $\{V_j\}_{j=1}^m$ provides a uniformly weighted projective resolution of the identity, we say that the associated analysis operator is a *uniform (m,l,k)-protocol*.

The ratio ml/k we shall refer to as the *redundancy* of the encoding.

The usual notion of a code is a linear subspace over some field. Here, we regard resolutions of the identity as codes because the analysis operator V maps an initial vector x in $\mathbb{F}^k$ into a subspace of $\mathbb{F}^{ml} = \bigoplus_{j=1}^m \mathbb{F}^l$. The vector $Vx \in \mathbb{F}^{ml}$ is an encoded version of x, which is transmitted to a receiver and then decoded by applying V^*. Among all possible left inverses of V, we have that V^* is the unique left inverse that minimizes both the operator norm and Hilbert–Schmidt norm.

3. Error bounds and optimal protocols

The problem we consider is that in the process of transmission some number of the packets $(V_j x)$ are lost, corrupted or just delayed for such a long time that one has to reconstruct x with what has been received.

Definition 7. Let $\mathbb{K} \subset \mathbb{J} = \{1, 2, \ldots, m\}$ be a subset of size $|\mathbb{K}| = q \in \mathbb{N}$. The *packet-loss operator* $E_{\mathbb{K}}$ on $\bigoplus_{j \in \mathbb{J}} \mathcal{K}$ is given by

$$E_{\mathbb{K}} : \bigoplus_{j=1}^m \mathcal{K} \to \bigoplus_{j=1}^m \mathcal{K}, \quad (E_{\mathbb{K}} y)_j = \begin{cases} y_j, & j \notin \mathbb{K}, \\ 0, & j \in \mathbb{K}. \end{cases}$$

We also denote $D_{\mathbb{K}} = I - E_{\mathbb{K}}$, and abuse the notation in the singleton case $\mathbb{K} = \{j\}$ to write the packet-loss operator as E_j.

The set of all $D_{\mathbb{K}}$ with $|\mathbb{K}| = q$ is denoted as $\mathcal{D}_q = \{I - E_{\mathbb{K}}\colon\ E_{\mathbb{K}}$ is packet-loss operator with $|\mathbb{K}| = q\}$.

The operator $E_{\mathbb{K}}$ can be thought of as erasing the coordinates $(V_j x)_{j \in \mathbb{K}}$ in the terminology of [8].

We recall from [2] that there are two methods by which one could attempt to reconstruct x. Either one computes a left inverse for $E_{\mathbb{K}} V$ or one continues to use the left inverse V^* for V and accepts that this "blind reconstruction" of x is only approximate.

If $E_{\mathbb{K}} V$ has a left inverse, then the left inverse of minimum norm is given by $T^{-1} W^*$ where $E_{\mathbb{K}} V = WT$ is the polar decomposition and $T = |E_{\mathbb{K}} V| = (V^* E_{\mathbb{K}} V)^{1/2}$. Thus, the minimum norm of a left inverse is given by $t_{\min}^{-1}$, where $t_{\min}$ denotes the smallest eigenvalue of T.

In the second alternative, the error in reconstructing x is given by

$$x - V^* E_{\mathbb{K}} V x = \bigl(I - T^2\bigr) x.$$

Thus, the norm of the error operator $V^*(I - E_{\mathbb{K}})V$ is $1 - t_{\min}^2$.

Hence, when a left inverse exists, the problems of minimizing the norm of a left inverse over all frames and of minimizing the norm of the error operator over all frames are both achieved by maximizing the minimal eigenvalue of T. A left inverse exists if and only if the norm of the reconstruction error operator $V^*(I - E_{\mathbb{K}})V$ is strictly less than 1.

Definition 8. Given an (m, l, k)-protocol V, we say that a packet loss operator $E_{\mathbb{K}}$, $\mathbb{K} \subset \{1, 2, \ldots, m\}$, is *correctible* if

$$\left\| V^*(I - E_{\mathbb{K}})V \right\| < 1.$$

For simplicity, we study exclusively the norms of error operators. It is left to the reader to convert the respective error bounds to those of the left inverses.

The main goal of this section is to characterize when the norms of these error operators are in some sense minimized for a given number of lost packets, independent of which packets are lost. Of course there are many ways that one could define optimality in this setting. Here, we only pursue one among several possibilities.

The performance measure that we introduce represents the maximal norm of an error operator given that q packets are set to zero.

Definition 9. Let $V : \mathcal{H} \to \bigoplus_{j \in \mathbb{J}} \mathcal{K}$ be an (m, l, k)-protocol. We denote the worst-case error when $q \in \mathbb{N}$ packets are lost as

$$e_q(V) = \max\left\{\|V^*V - V^*E_{\mathbb{K}}V\|:\ \mathbb{K} \subset \mathbb{J},\ |\mathbb{K}| = q\right\},$$

where $\|\cdot\|$ denotes the operator norm.

Since the set $\mathcal{V}(m, l, k)$ of all (m, l, k)-protocols is a compact set, the value

$$e_1(m, l, k) = \inf\left\{e_1(V):\ V \in \mathcal{V}(m, l, k)\right\}$$

is attained and we define the set of 1-*loss optimal protocols* to be the nonempty compact set $\mathcal{V}_1(m, l, k)$ where this infimum is attained, i.e.,

$$\mathcal{V}_1(m, l, k) = \left\{V \in \mathcal{V}(m, l, k):\ e_1(V) = e_1(m, l, k)\right\}.$$

Proceeding inductively, we now set for $2 \leqslant q \leqslant m$,

$$e_q(m, l, k) = \inf\left\{e_q(V):\ V \in \mathcal{V}_{q-1}(m, l, k)\right\}$$

and define the *optimal q-loss protocols* to be the nonempty compact subset $\mathcal{V}_q(m, l, k)$ of $\mathcal{V}_{q-1}(m, l, k)$ where this infimum is attained.

We define an equivalence relation among protocols in such a way that equivalent protocols have equal performance under any number q of lost packets.

Definition 10. We say that two (m, l, k)-protocols with analysis operators V and W are *equivalent* if there is a permutation π among the indices $\mathbb{J} = \{1, 2, \ldots, m\}$, a set of m unitary operators $\{U_j\}_{j=1}^m$ on $\mathcal{K}$ and a unitary Y on $\mathcal{H}$ such that for all $j \in \mathbb{J}$, the coordinate operators $\{V_j\}$ and $\{W_j\}$ are related by

$$U_j V_j Y = W_{\pi(j)}.$$

Remark 11. If V and W belong to equivalent protocols, then $UVV^*U^* = WW^*$ for an appropriate unitary U on $\bigoplus_j \mathcal{K}$, that is the product of $\Pi \otimes I$ and a block-diagonal unitary $\operatorname{diag}(U_j)$, where Π is a permutation matrix on $\mathbb{F}^m$ and $\{U_j\}$ a set of unitaries on $\mathcal{K}$. We note that for any packet loss operator E, we have $E = U^*E^2U$. Consequently, denoting $D = I - E$, the norms $\|V^*DV\| = \|V^*U^*D^2UV\| = \|DUVV^*U^*D\| = \|DWW^*D\| = \|W^*DW\|$ which implies that the operator norm of the reconstruction error V^*DV is the same for all protocols in one equivalence class.

3.1. Optimality for one lost packet

Lemma 12. *If the coordinate operators $\{V_j : \mathcal{H} \to \mathcal{K}\}$ belong to an (m, l, k)-protocol on a Hilbert space $\mathcal{H}$, then*

$$\max_j \|V_j^* V_j\| \geqslant \frac{k}{ml}$$

and equality holds if and only if for all $j \in \{1, 2, \ldots, m\}$ we have $V_j^* V_j = \frac{k}{ml} P_j$, where P_j is a self-adjoint rank-l projection operator.

Proof. We begin by comparing the maximum with the average of the operator norms,

$$\max_j \|V_j^* V_j\| \geqslant \frac{1}{m}\sum_{j=1}^m \|V_j^* V_j\|.$$

Denoting P_j the self-adjoint projection onto the range of $V_j^* V_j$, we have that $\|V_j^* V_j\| P_j \geqslant V_j^* V_j$ and by taking the trace, $l\|V_j^* V_j\| \geqslant \operatorname{tr}[V_j^* V_j]$. So we can continue estimating

$$\max_j \|V_j^* V_j\| \geqslant \frac{1}{ml}\sum_{j=1}^m \operatorname{tr}[V_j^* V_j] = \frac{k}{ml}.$$

If equality holds, then for each j, $l\|V_j^* V_j\| = \operatorname{tr}[V_j^* V_j]$, which means that each $V_j^* V_j$ is rank l and has only one nonzero eigenvalue $\|V_j^* V_j\|$. Dividing by this eigenvalue turns $V_j^* V_j$ into the self-adjoint projection P_j. □

Theorem 13. *Let $m, l, k \in \mathbb{N}$, and let $V : \mathcal{H} \to \bigoplus_{j=1}^m \mathcal{K}$ be an (m, l, k)-protocol. Then*

$$e_1(V) \geqslant \frac{k}{ml}$$

and equality holds if and only if the coordinate operators $\{V_j : \mathcal{H} \to \mathcal{K}\}_{j=1}^m$ satisfy that for all $j \in \{1, 2, \ldots, m\}$,

$$V_j^* V_j = \frac{k}{ml} P_j$$

with self-adjoint rank-l projections $\{P_j\}_{j=1}^m$ on $\mathcal{H}$.

Proof. For a given j, we note $V^*V - V^*E_jV = V^*D_jV$ and $\|V^*D_jV\| = \|D_jVV^*D_j\| = \|V_jV_j^*\| = \|V_j^*V_j\|$. So the minimized quantity is $e_1(V) = \max_j \|V^*D_jV\| = \max_j \|V_j^*V_j\|$.

Now the preceding lemma gives the claimed lower bound for $e_1(V)$ and characterizes for which V it is attained. □

The consequence of this characterization of optimality is that if there exist m uniformly weighted rank-l projections resolving the identity on a Hilbert space $\mathcal{H}$ of dimension k, then the associated uniform (m, l, k)-protocols are precisely the 1-loss optimal ones. Since the rank of all projections is required to be the same, these protocols are a special case of what has also been referred to as Parseval frames for subspaces [4] or Parseval fusion frames [5].

3.2. Optimality for two lost packets

We now turn to the case of two lost packets. The following lemma reduces to a bound that goes back to Welch in the case of rank $l = 1$ [20]. The present generalization was also derived in a slightly different form in the context of quantum communication [1]. We use the opportunity to give a simpler, more geometry-oriented proof.

We abbreviate

$$c_{m,l,k} = \sqrt{\frac{k(ml-k)}{m^2 l^2 (m-1)}}.$$

Lemma 14. *If $\{V_j\}_{j=1}^m$, $m \geqslant 2$, is a family of uniformly weighted rank-l coordinate operators of a projective resolution of the identity on a Hilbert space $\mathcal{H}$ of dimension k, then*

$$\max_{i \neq j} \|V_i V_j^*\| \geqslant c_{m,l,k}$$

and equality holds if and only if for all $i \neq j$, $V_i V_j^* = c_{m,l,k} Q_{i,j}$ with $Q_{i,j}$ a unitary on $\mathcal{K}$.

Proof. From the fact that VV^* is a rank-k projection, we obtain

$$k = \operatorname{tr}[VV^*] = \operatorname{tr}[VV^*VV^*] = \sum_{i,j} \operatorname{tr}\bigl[|V_i V_j^*|^2\bigr].$$

Using the same strategy as in Lemma 12, we estimate $l\|V_iV_j^*\|^2 \geqslant \mathrm{tr}[|V_iV_j^*|^2]$. In combination with the assumption $\|V_iV_i^*\| = \|V_i^*V_i\| = \frac{k}{ml}$, we have

$$\sum_{i,j=1}^{m} l\|V_iV_j^*\|^2 = \frac{k^2}{ml} + \sum_{i\neq j} l\|V_iV_j^*\|^2 \geqslant k.$$

Comparing the maximum with the average norm gives

$$m(m-1)\max_{i\neq j}\|V_iV_j^*\|^2 \geqslant \sum_{i\neq j}\|V_iV_j^*\|^2 \geqslant \frac{k}{l} - \frac{k^2}{ml^2}.$$

If equality holds, then for all $i, j \in \{1, 2, \ldots, m\}$, $\mathrm{tr}[|V_iV_j^*|^2] = l\|V_iV_j^*\|^2$ and $|V_iV_j^*| = \sqrt{V_iV_j^*V_jV_i^*}$ has only one eigenvalue $c_{m,l,k} = \|V_iV_j^*\|$, so by polar decomposition $V_iV_j^* = c_{m,l,k}Q_{i,j}$ with $Q_{i,j}$ a unitary on $\mathcal{K}$. Conversely, assuming for each $i \neq j$, $V_iV_j^* = c_{m,l,k}Q_{i,j}$ with a unitary $Q_{i,j}$ on $\mathcal{K}$, then $\|V_iV_j^*\| = c_{m,l,k}$. □

Theorem 15. *Let $m, l, k \in \mathbb{N}$. If $V : d\mathcal{H} \to \bigoplus_{j=1}^{m}\mathcal{K}$ is a uniform (m, l, k)-protocol, then if $m \geqslant 2$,*

$$e_2(V) \geqslant \frac{k}{ml} + c_{m,l,k}$$

and equality holds if and only if for each pair $i, j \in \{1, 2, \ldots, m\}$, $i \neq j$, we have $V_iV_j^* = c_{m,l,k}Q_{i,j}$ with $Q_{i,j}$ a unitary on $\mathcal{K}$.

Proof. We recall that the coordinate operators of a uniform (m, l, k)-protocol satisfy $V_jV_j^* = \frac{k}{ml}P_j$ and $\sum_{j=1}^{m}\frac{k}{ml} \times P_j = I$, with self-adjoint projection operators $\{P_j\}$ on $\mathcal{H}$.

The worst-case reconstruction error for two lost packets is

$$e_2(V) = \max_{i\neq j}\bigl\|V^*(D_i + D_j)V\bigr\| = \max_{i\neq j}\bigl\|(D_i + D_j)VV^*(D_i + D_j)\bigr\|.$$

The matrices whose norm is maximized are nonnegative, so the norm is equal to the largest eigenvalue. Using the Cauchy–Schwarz inequality, we see for any $x \in \bigoplus_j \mathcal{K}$,

$$\begin{aligned}\bigl\langle (D_i + D_j)VV^*(D_i + D_j)x, x\bigr\rangle &\leqslant \frac{k}{ml}\bigl(\|x_i\|^2 + \|x_j\|^2\bigr) + 2\mathrm{Re}\langle V_iV_j^*x_j, x_i\rangle_{\mathcal{K}}\\ &\leqslant \frac{k}{ml}\bigl(\|x_i\|^2 + \|x_j\|^2\bigr) + 2\|V_iV_j^*\|\|x_i\|\|x_j\|\end{aligned}$$

with equality if $V_iV_j^*x_j$ is collinear with x_i and $\langle V_iV_j^*x_j, x_i\rangle \geqslant 0$. The right-hand side is maximized subject to the constraint $\|x_i\|^2 + \|x_j\|^2 = 1$ when $\|x_i\| = \|x_j\| = \frac{1}{\sqrt{2}}$. Therefore,

$$\left\|\begin{pmatrix} \frac{k}{ml}I & V_iV_j^* \\ V_jV_i^* & \frac{k}{ml}I \end{pmatrix}\right\| = \frac{k}{ml} + \|V_iV_j^*\|$$

and maximizing over all $i \neq j$ gives

$$e_2(V) = \frac{k}{ml} + \max_{i\neq j}\|V_iV_j^*\|.$$

Now the preceding lemma implies that $e_2(V)$ is bounded below by $\frac{k}{ml} + c_{m,l,k}$ and the bound is saturated if and only if all $\{V_iV_j^*\}_{i\neq j}$ satisfy $V_iV_j^* = c_{m,l,k}Q_{i,j}$ with a set of unitaries $\{Q_{i,j}\}_{i\neq j}$ on $\mathcal{K}$. □

The case when this bound is saturated describes a set of protocols we investigate further.

Definition 16. We call a linear map $V : \mathcal{H} \to \bigoplus_{j=1}^{m}\mathcal{K}$ a *two-uniform (m, l, k)-protocol* provided that the coordinate operators of V are uniform and in addition there is a constant $c > 0$ such that $\|V^*(I - E)V\| = c$ for all two-packet loss operators E.

Remark 17. Two-uniform (m,l,k)-protocols can be characterized in geometric terms by the subspaces onto which the self-adjoint projections $\{P_j = \frac{ml}{k} V_j^* V_j\}$ map the Hilbert space $\mathcal{H}$.

The fact that for $i \neq j$, $V_i V_j^* = c_{m,l,k} Q_{i,j}$ with a unitary $Q_{i,j}$ on $\mathcal{K}$ means that for every $x \in \mathcal{K}$, $\|V_i V_j^* x\| = c_{m,l,k}\|x\|$. However, V_i^* and V_j^* are isometries, so for any $y \in \mathcal{H}$ in the range of V_j^*, we have $\|V_i^* V_i y\| = c_{m,l,k}\|y\|$. This means, for all $i \neq j$, projecting any vector in the range of P_j onto the range of P_i changes its length by the scalar multiple $c_{m,l,k}$. Such a family of subspaces is called *equi-isoclinic* [6,14].

Definition 18. Given a two-uniform (m,l,k)-protocol $V : \mathcal{H} \to \bigoplus_{j=1}^m \mathcal{K}$, then $VV^* = aI + c_{m,l,k} Q$ is a projection on $\bigoplus_j \mathcal{K}$, where $a = k/ml$, $c_{m,l,k}$ is the lower bound in Lemma 14, and $Q = (Q_{i,j})_{i,j=1}^m$ is a self-adjoint matrix containing the zero operator $Q_{i,i} = 0$ on $\mathcal{K}$ for all $i \in \{1,2,\ldots,m\}$ and unitaries $\{Q_{i,j}\}$ on $\mathcal{K}$ for off-diagonal entries indexed by $i \neq j$. We call this self-adjoint matrix of operators Q the *signature matrix* of V.

A key result about signature matrices is the following theorem which generalizes a result in [10].

Theorem 19. *Let Q be a self-adjoint operator on $\bigoplus_{j=1}^m \mathcal{K}$, $m \geqslant 2$, with diagonal components $Q_{i,i} = 0$ for all $i \in \{1,2,\ldots,m\}$ and unitaries $\{Q_{i,j}\}$ for all $i \neq j$. Then Q is the signature matrix of a two-uniform (m,l,k)-protocol $V : \mathcal{H} \to \bigoplus_{j=1}^m \mathcal{K}$ with $\dim(\mathcal{H}) = k < ml$ if and only if*

$$Q^2 = (m-1)I + \mu Q$$

for $\mu = (1 - 2k/ml)/c_{m,l,k}$.

Proof. We denote $a = \frac{k}{ml}$. If $VV^* = a\,I + c_{m,l,k} Q$ is a projection, then

$$a^2 I + 2a\, c_{m,l,k} Q + c_{m,l,k}^2 Q^2 = a\, I + c_{m,l,k} Q.$$

Solving this for Q^2 yields

$$Q^2 = \frac{a - a^2}{c_{m,l,k}^2} I + \frac{1-2a}{c_{m,l,k}} Q = (m-1)I + \mu Q.$$

Conversely, assuming Q satisfies the preceding quadratic equation. Then it has only two eigenvalues ρ_1 and ρ_2. Reversing the steps of the above calculation, one can verify that $a + c_{m,l,k}\rho_n \in \{0,1\}$ for $n = 1$ and $n = 2$. □

The above theorem reduces the construction of two-uniform (m,l,k)-protocols to producing matrices Q satisfying the appropriate equations. We only give the simplest examples.

Example 20. Let J_m denote the $m \times m$ matrix all of whose entries are the identity I on $\mathcal{K}$. Then $Q = J_m - I$ satisfies $Q^2 = J_m^2 - 2J_m + I = (m-2)J_n + I = (m-1)I + (m-2)Q$ and so, by our above formulas and the rank of the largest eigenvalue of Q, we obtain $k = l$ and $\mu = m-2$, yielding a rather uninteresting two-uniform (m,l,l)-protocol for $\mathbb{F}^l$.

However, $Q = I - J_m$ is also a signature matrix with $\mu = 2 - m$, $k = l(m-1)$, which shows that for each $k = l(m-1)$ there exists a two-uniform $(m,l,l(m-1))$-protocol.

The next example describes how to construct the preceding example and other two-uniform (m,l,k)-protocols in case l divides k using a tensor construction and two-uniform frames.

Example 21. Let $\mathbb{F}^{sl} = \mathbb{F}^s \otimes \mathbb{F}^l$ with $s, l \in \mathbb{N}$, and assume Q is a signature matrix of a two-uniform $(m,1,s)$-protocol V. In this case, $VV^* = \frac{s}{m} I + c_{m,1,s} Q$ is the Grammian of a two-uniform frame [2,10].

We observe that $|Q_{i,j}| = \|Q_{i,j} I\|$ and $c_{m,1,s} = c_{m,l,sl}$, so the matrix $Q \otimes I$ on $\bigoplus_{j=1}^m \mathbb{F}^l$ is the signature matrix of the two-uniform (m,l,sl)-protocol $V \otimes I$.

A slightly more general construction, which is possible when $\mathbb{F}^k = \bigoplus_{n=1}^l \mathbb{F}^s$ and $m, s \in \mathbb{N}$ allow two-uniform $(m,1,s)$-protocols, is to take l such signature matrices $\{Q^{(1)}, Q^{(2)}, \ldots, Q^{(l)}\}$ and combine them as $\bigoplus_{n=1}^l Q^{(n)}$. Then performing the canonical shuffle between $\mathbb{F}^l$ and $\mathbb{F}^m$ yields the signature matrix of a two-uniform (m, l, k)-protocol. Conversely, if all off-diagonal entries $Q_{i,j} = V_i V_j^*$ in a signature matrix of a two-uniform (m, l, k)-protocol commute, then by choosing an appropriate basis of $\mathcal{K}$, all of these entries can be diagonalized simultaneously and after the canonical shuffle, Q becomes a direct sum $\bigoplus_{n=1}^{l} Q^{(n)}$ of signature matrices belonging to two-uniform $(m, 1, s)$-protocols.

Lemmens and Seidel [14] describe a more sophisticated construction to obtain examples of real equi-isoclinic subspaces and thus of real signature matrices. Godsil and Hensel [6] show how to obtain such subspaces from distance regular antipodal covers of the complete graph. It is an open problem to find a graph theoretic characterization of equivalence classes of two-uniform protocols for real Hilbert spaces. Even less seems to be known about generic constructions and an analogue of the graph-theoretic characterization of two-uniform protocols in the complex case.

3.3. Optimality for three lost packets

Next, for given dimensions m, l and $k \in \mathbb{N}$, we want to minimize the worst-case Euclidean reconstruction error for three lost packets among two-uniform (m, l, k)-protocols.

For any three-element subset of indices $\mathbb{K} = \{h, i, j\} \subset \mathbb{J} = \{1, 2, \ldots, m\}$, we denote the compression of an $m \times m$ (block) matrix M to the corresponding rows and columns as

$$[M]_{\mathbb{K}} = \begin{pmatrix} M_{h,h} & M_{h,i} & M_{h,j} \\ M_{i,h} & M_{i,i} & M_{i,j} \\ M_{j,h} & M_{j,i} & M_{j,j} \end{pmatrix}.$$

Lemma 22. *Given a two-uniform (m, l, k)-protocol $V : \mathcal{H} \to \bigoplus_{j=1}^{m} \mathcal{K}$, $k = \dim \mathcal{H} < ml$, and its signature matrix Q, and a three-element subset of indices $\mathbb{K} = \{h, i, j\} \subset \mathbb{J}$, then the maximal eigenvalue of the matrix $[Q]_{\mathbb{K}}$ is $2\cos(\theta/3)$, where $\theta \in [-\pi, \pi]$ is determined from the maximal eigenvalue $\lambda = 2\cos\theta$ of the self-adjoint operator $Q_{h,i} Q_{i,j} Q_{j,h} + Q_{h,j} Q_{j,i} Q_{i,h}$ on $\mathcal{K}$.*

Proof. Conjugating with the block-diagonal unitary $\mathrm{diag}(I, Q_{i,h}, Q_{j,h})$, we transform $[Q]_{\mathbb{K}}$ into the unitarily equivalent matrix

$$[\tilde{Q}]_{\mathbb{K}} = \begin{pmatrix} 0 & I & I \\ I & 0 & Q_{h,i} Q_{i,j} Q_{j,h} \\ I & Q_{h,j} Q_{j,i} Q_{i,h} & 0 \end{pmatrix}.$$

Now we change to the eigenbasis of the unitary $U = Q_{h,i} Q_{i,j} Q_{j,h}$ in $\mathcal{K}$ to diagonalize the remaining blocks and observe that the resulting matrix is a direct sum of 3×3 matrices of the form $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & e^{i\theta} \\ 1 & e^{-i\theta} & 0 \end{pmatrix}$, with an angle $\theta \in [-\pi, \pi]$. The largest eigenvalue of such a matrix is $2\cos(\theta/3)$, where by the monotonicity of the cosine the angle θ also gives the maximal eigenvalue $2\cos\theta$ of the self-adjoint operator $U + U^*$. By unitary equivalence, the largest eigenvalue of both $[\tilde{Q}]_{\mathbb{K}}$ and of $[Q]_{\mathbb{K}}$ is $2\cos(\theta/3)$. $\square$

Lemma 23. *Let Q be the signature matrix of a two-uniform (m, l, k)-protocol V. If $m \geqslant 3$ and $k < ml$, then*

$$\sum_{h,i,j=1}^{m} \mathrm{tr}[Q_{h,i} Q_{i,j} Q_{j,h}] = \frac{(m-1)(ml-2k)}{c_{m,l,k}}.$$

Proof. We identify the sum as the trace

$$\sum_{h,i,j=1}^{m} \mathrm{tr}[Q_{h,i} Q_{i,j} Q_{j,h}] = \mathrm{tr}\big[Q^3\big].$$

Using the equation $Q^2 = (m-1)I + \mu Q$ twice, we can reduce the power

$$Q^3 = \mu(m-1)I + \big(m - 1 + \mu^2\big)Q.$$

Inserting this expression in $\mathrm{tr}[Q^3]$ gives together with $\mathrm{tr}[Q]=0$ the claimed value

$$\mathrm{tr}\big[Q^3\big]=ml(m-1)\mu=(m-1)\frac{ml-2k}{c_{m,l,k}}. \qquad \square$$

The following theorem gives a lower bound for e_3 among all two-uniform (m,l,k)-protocols. It generalizes earlier results [2, Section 5.2], and is of interest even for $l=1$ in the case of a complex Hilbert space $\mathcal{H}$. If $\mathcal{H}$ is a real Hilbert space and $\mathcal{K}=\mathbb{R}$, then it can be reduced to a known statement [2, Section 5.2].

Theorem 24. *Let $m,l,k\in\mathbb{N}$, $m\geqslant 3$ and $k<ml$. Let $V:\mathcal{H}\to\bigoplus_{j=1}^{m}\mathcal{K}$ be a two-uniform (m,l,k)-protocol. Then*

$$e_3(V)\geqslant\frac{k}{ml}+2c_{m,l,k}\cos(\theta/3),$$

where $\theta\in[-\pi,\pi]$ observes

$$\cos\theta=\frac{ml-2k}{ml(m-2)c_{m,l,k}}.$$

When $k<ml$, the protocol V saturates the lower bound for e_3 if and only if the signature matrix Q of V satisfies that for all $\{h,i,j\}\subset\{1,2,\ldots,m\}$, the largest eigenvalue of $Q_{h,i}Q_{i,j}Q_{j,h}+Q_{h,j}Q_{j,i}Q_{i,h}$ is $2\cos(\theta)$.

Proof. The worst-case norm of the reconstruction error $\|DVV^*D\|$ among all $D\in\mathcal{D}_3$, is the largest operator norm among the compressions of VV^* to three rows and columns. If $k=ml$, then $c_{m,l,k}=0$ and $e_3(V)=1$ because $VV^*=I$.

So for the remainder of the proof, we can assume $k<ml$. Since VV^* is positive, finding the worst-case error $e_3(V)$ amounts to finding the largest eigenvalue among the 3×3 compressions of the signature matrix Q. By Lemma 22, the maximal eigenvalue among the matrices $[Q]_{\mathbb{K}}$, $\mathbb{K}\subset\{1,2,\ldots,m\}$ with $|\mathbb{K}|=3$ is the maximum of $2\cos(\theta/3)$ among the set of all eigenvalues $e^{i\theta}$ of products $Q_{h,i}Q_{i,j}Q_{j,h}$ indexed by three-element subsets $\{h,i,j\}\subset\{1,2,\ldots,m\}$.

To minimize the worst-case norm of the reconstruction error, this maximum must be optimally suppressed. By the monotonicity of the cosine, this is achieved by minimizing the maximal eigenvalue occurring among all $\{S_{h,i,j}=Q_{h,i}Q_{i,j}Q_{j,h}+Q_{h,j}Q_{j,i}Q_{i,h}\colon h\neq i\neq j\neq h\}$. Lemma 23 states that

$$\sum_{h,i,j=1}^{m}\mathrm{tr}[S_{h,i,j}]=\frac{2m-2}{c_{m,l,k}}(ml-2k),$$

where we define $S_{h,i,j}=0$ whenever any two indices are equal. For each choice $\{h,i,j\}\subset\{1,2,\ldots,m\}$, let $\lambda_{h,i,j}$ denote the maximal eigenvalue of $S_{h,i,j}$. Then by $l\lambda_{h,i,j}\geqslant\mathrm{tr}[S_{h,i,j}]$, we have

$$m(m-1)(m-2)\max_{h\neq i\neq j\neq h}l\lambda_{h,i,j}\geqslant\sum_{h,i,j=1}^{m}\mathrm{tr}[S_{h,i,j}]$$

and thus

$$\max_{h\neq i\neq j\neq h}\lambda_{h,i,j}\geqslant\frac{2ml-4k}{ml(m-2)c_{m,l,k}}.$$

Now setting $\cos\theta=\frac{ml-2k}{ml(m-2)c_{m,l,k}}$ and using Lemma 22 gives the desired estimate. $\square$

For examples of cyclic two-uniform protocols with $l=1$, also called cyclic two-uniform frames, that have $\max_{h,i,j}\lambda_{h,i,j}<2$, see [13]. However, at present the only protocol known to the author that saturates the lower bound for e_3 with $e_3(V)<1$ is the two-uniform (m,l,l)-protocol.

Example 25. For the two-uniform (m,l,l)-protocol of Example 20, we calculate $c_{m,l,l}=\frac{1}{m}$ and notice that all 3×3-compressions of the signature matrix are identical with $\cos\theta=1$. Consequently, $e_3(m,l,l)=\frac{3}{m}$.

On the other hand, for the two-uniform $(m,l,(m-1)l)$-protocol we also have $c_{m,l,(m-1)l}=\frac{1}{m}$, but $\cos\theta=-1$ and $e_3(m,l,(m-1)l)=1$.

4. Correctibility of two-uniform protocols for a higher number of lost packets

If the largest eigenvalue among all $\{S_{h,i,j}\colon h \neq i \neq j \neq h\}$ is two for the signature matrix of a two-uniform (m,l,k)-protocol, then this protocol maximizes the worst-case norm of the reconstruction error for $q = 3$ lost packets. We characterize the analogue of this situation for higher values of q.

Definition 26. We say that a two-uniform (m,l,k)-protocol $V : \mathcal{H} \to \bigoplus_{j=1}^{m} \mathcal{K}$ with signature matrix Q has a nontrivial *q-packet covariant vector* $x \in (\bigoplus_j \mathcal{K}) \setminus \{0\}$ if there is a subset $\mathbb{K} \subset \{1, 2, \ldots, m\}$ of size $|\mathbb{K}| = q$ such that $x_j = 0$ if $j \notin \mathbb{K}$ and $Q_{i,j} x_j = x_i$ for all $i, j \in \mathbb{K}$ with $i \neq j$.

If $\mathcal{H}$ is a real Hilbert space and $\mathcal{K} = \mathbb{R}$, then the "unitaries" $\{Q_{i,j}\}$ are scalars ± 1 and the presence of a covariant vector amounts to partitioning $\mathbb{K}$ into two subsets, such that $Q_{i,j} = -1$ whenever i and j belong to different subsets. This can be restated in graph-theoretic terminology, which is the basis for the derivation of error bounds [2] in this special case. Here, we derive an analogous result for packet encoding.

Theorem 27. *Let $m, l, k, q \in \mathbb{N}$. If V is a two-uniform (m,l,k)-protocol with signature matrix Q then*

$$e_q(V) \leqslant \frac{k}{ml} + (q-1)c_{m,l,k}$$

and equality holds for $k < ml$ if and only if V has a nontrivial q-packet covariant vector.

Proof. If $d = ml$, then $c_{m,l,k} = 0$, VV^* is the identity, and $e_q(V) = \|DVV^*D\| = 1$.

Now suppose $d < ml$, which implies $c_{m,l,k} > 0$. Since VV^* is a positive operator on $\bigoplus_j \mathcal{K}$, so is any compression to rows and columns corresponding to lost packets. Thus, maximizing the operator norm of $\|DVV^*D\| = \|D(aI + c_{m,l,k}Q)D\|$ for all $D \in \mathcal{D}_q$ is achieved by maximizing the inner product $\langle Qx, x\rangle$ in $\bigoplus_j \mathcal{K}$ over the unit vectors $x \in \bigoplus_j \mathcal{K}$ for which there is a set $\mathbb{K} \subset \{1, 2, \ldots, m\}$ of size $|\mathbb{K}| = q$ such that $x_j = 0$ for all $j \notin K$.

Such inner products can be written as

$$\langle Qx, x\rangle = \sum_{\substack{i,j \in \mathbb{K},\\ i \neq j}} \langle Q_{i,j} x_j, x_i\rangle_{\mathcal{K}}.$$

We estimate the right-hand side using the Cauchy–Schwarz inequality for each inner product on $\mathcal{K}$

$$\langle Qx, x\rangle \leqslant \sum_{\substack{i,j \in \mathbb{K},\\ i \neq j}} \|x_i\|\|x_j\|.$$

This upper bound is maximized subject to the constraint $\sum_j \|x_j\|^2 = 1$ when $\|x_j\| = \frac{1}{\sqrt{q}}$ for all $j \in \mathbb{K}$. Computing the maximum gives

$$\langle Qx, x\rangle \leqslant q - 1.$$

Equality holds in this estimate if it holds for all applications of the Cauchy–Schwarz inequality. This means $Q_{i,j}x_j$ and x_i are collinear, which implies by the unitarity of $Q_{i,j}$ that $Q_{i,j}x_j = x_i$. Therefore, if the inequality is saturated then the vector x is a nontrivial q-packet covariant vector. Conversely, if there is such a q-packet covariant vector x, then all norms of its nonzero components are equal and we can change x by a scalar multiple to normalize $\|x_j\| = \frac{1}{\sqrt{q}}$. Then x satisfies $\langle Qx, x\rangle = q - 1$. □

We use this theorem to derive a sufficient condition for correctibility of packet losses.

Corollary 28. *If V is a two-uniform (m,l,k)-protocol, $k < ml$, then any q-packet loss operator is correctible if $1 \leqslant q < 1 + \sqrt{\frac{(m-1)(ml-k)}{k}}$. Moreover, if V does not have a nontrivial q-packet covariant vector, then we can allow $1 \leqslant q \leqslant 1 + \sqrt{\frac{(m-1)(ml-k)}{k}}$ and still retain correctibility.*

Proof. We recall that a packet loss operator E is called correctible if $\|V(I-E)V^*\| = \|(I-E)VV^*(I-E)\| < 1$. To prove the first part of the corollary, we need to show

$$e_q(V) \leqslant \frac{k}{ml} + (q-1)c_{m,l,k} < 1.$$

By the assumption and the monotonicity of the square function for positive arguments,

$$(q-1)^2 < \frac{(m-1)(ml-k)}{k}.$$

This implies

$$\frac{k}{m-1} < \frac{ml-k}{(q-1)^2}$$

and in turn

$$\frac{k(ml-k)}{m^2l^2(m-1)} < \frac{1}{(q-1)^2}\left(1-\frac{k}{ml}\right)^2.$$

Now taking the square root, multiplying both sides by $q-1$ and adding k/ml gives the desired inequality.

In case there is no q-packet covariant vector, we can relax the assumption to

$$(q-1)^2 \leqslant \frac{(m-1)(ml-k)}{k},$$

since we know the upper bound for $e_q(V)$ in the preceding theorem is not saturated, thus

$$e_q(V) < \frac{k}{ml} + (q-1)c_{m,l,k} \leqslant 1. \qquad \square$$

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